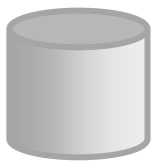
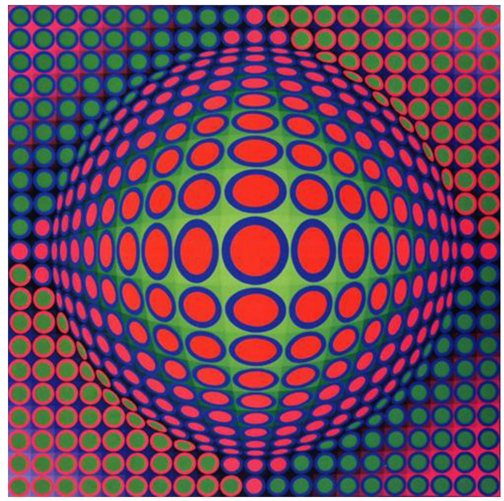
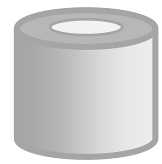
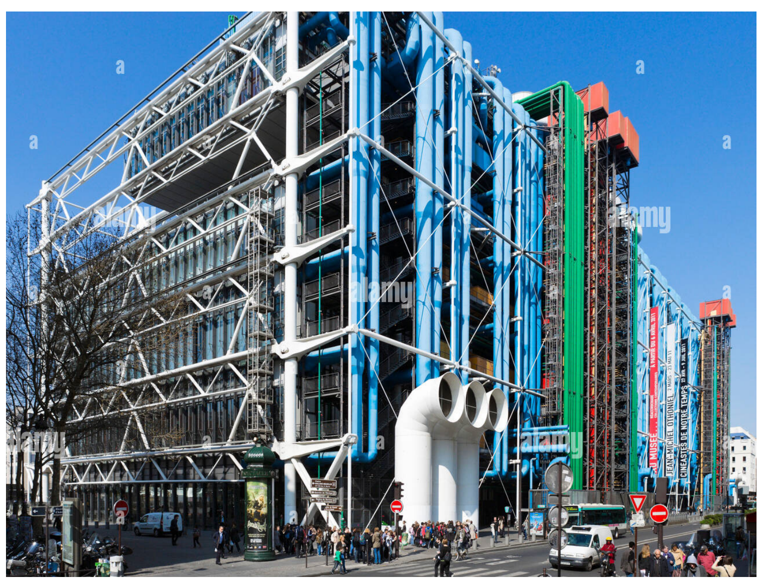
VIII.3.3. HEAT EQUATION IN CYLINDRICAL COORDINATES









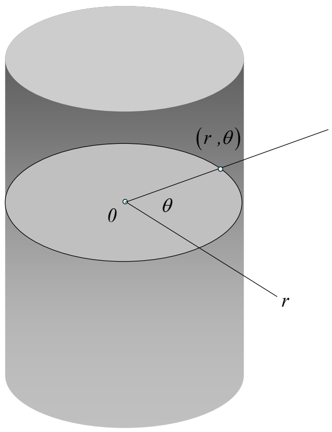
VIII.3.3.4 HEAT EQUATION in Cylindrical Coordinates

$u(r,\theta,z,t)$

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}+\frac{g}{k}=\frac{1}{\alpha}\frac{\partial u}{\partial t}$$

1) Long cylinder $\left(\frac{\partial u}{\partial z}=0\right)$:

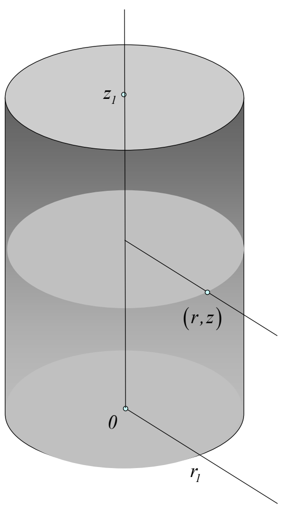
$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{g}{k}=\frac{1}{\alpha}\frac{\partial u}{\partial t}$$



$u(r,\theta,t)$

2) Short cylinder with angular symmetry $\left(\frac{\partial u}{\partial \theta}=0\right)$:

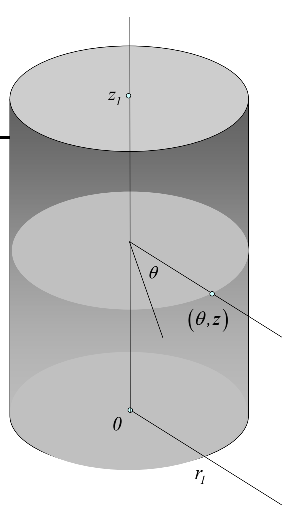
$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}+\frac{g}{k}=\frac{1}{\alpha}\frac{\partial u}{\partial t}$$



$u(r,z,t)$

3) Cylindrical surface of fixed radius r_1 $\left(\frac{\partial u}{\partial r}=0\right)$:

$$\frac{1}{r_1^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}+\frac{g}{k}=\frac{1}{\alpha}\frac{\partial u}{\partial t}$$



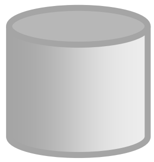
$u(\theta,z,t)$

Thin-wall cylindrical pipe

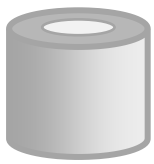
$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

THE HEAT EQUATION

Cylindrical Coordinates



$[u]_S = f$



$[u]_S = f$

$$\nabla^2 u + F(r,\theta,z) = \frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad t > 0$$

$$[u]_S = f$$

$$[u]_{t=t_0} = u_0$$

Domain:

solid cylinder

$(r,\theta,z) \in [0,r_1) \times [-\pi,\pi] \times (0,L) \subset \mathbb{R}^3$

hollow cylinder

$(r,\theta,z) \in (r_1,r_2) \times [-\pi,\pi] \times (0,L) \subset \mathbb{R}^3$

$$u(r,\theta,z,t) = u_s(r,\theta,z) + U(r,\theta,z,t)$$

$u_s(r,\theta,z)$ $\qquad$ $U(r,\theta,z,t)$

STEADY STATE PROBLEM - PE

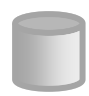
$$\nabla^2 u + F(r,\theta,z) = 0$$

$[u_s]_S = f$

TRANSIENT PROBLEM - HE

$$\nabla^2 U \equiv \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2} = \frac{1}{\alpha}\frac{\partial U}{\partial t}$$

$[U]_S = 0$



initial condition

$[U]_{t=t_0} = u_0 - u_s = U_0(r,\theta,z)$

supplemental eigenvalue problems

$\Theta'' = \eta\Theta$ $\Theta(\theta+2\pi) = \Theta(\theta)$	$\overset{SLP}{\Rightarrow}$	$\Theta_n'' = -n^2\Theta_n \qquad n = 0,1,2,\ldots$ $\eta_0 = 0 \qquad \Theta_0(\theta) = 1$ $\eta_n = -n^2 \qquad \Theta_n(\theta) = a_n\cos(n\theta) + b_n\sin(n\theta)$
$R'' + \frac{1}{r}R' - \frac{n^2}{r^2}R = \mu R$ $R(0) < \infty$ $[R(r_1)] = 0$	$\overset{SLP}{\Rightarrow}$	$R_{nm}'' + \frac{1}{r}R_{nm}' - \frac{n^2}{r^2}R_{nm} = -\lambda_{nm}^2 R$ $\mu_{nm} = -\lambda_{nm}^2$ $r^2 R_{nm}'' + rR_{nm}' + (r^2\lambda_{nm}^2 - n^2)R_{nm} = 0$ $R_{nm}(r) = J_n(\lambda_{nm} r) \qquad n = 0,1,2,\ldots \quad m = (0),1,2,\ldots$
$Z'' = \gamma Z$ $[Z]_{z=0} = 0$ $[Z]_{z=K} = 0$	$\overset{SLP}{\Rightarrow}$	$Z_k'' = -\omega_k^2 Z_k$ $\gamma_k = -\omega_k^2$ $Z_k(z) \qquad k = (0),1,2,\ldots$

SEPARATION OF VARIABLES

$$U(r,\theta,z,t) = \Phi(r,\theta,z)\,T(t)$$

HELMHOLTZ EQUATION

$$\frac{\nabla^2\Phi}{\Phi} = \beta$$

$$\Phi(r,\theta,z) = R(r)\Theta(\theta)Z(z)$$

$$\beta_{nmk} = -\left(\lambda_{nm}^2 + \omega_k^2\right)$$

$$R_{nm},\ \Theta_n,\ Z_k$$

$$\frac{1}{\alpha}\frac{T'}{T} = \beta$$

$$T_{nmk}(t) = e^{-\alpha\left(\lambda_{nm}^2 + \omega_k^2\right)t}$$

TRANSIENT SOLUTION

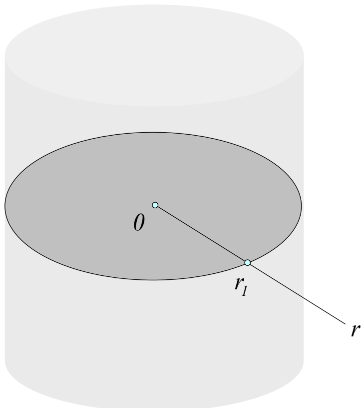
$$U(r,\theta,z,t) = \Phi(r,\theta,z)T(t)$$

see p.654 for the case of solid cylinder, and p.658 for the case of hollow cylinder

SOLUTION OF IBVP

$$u(r,\theta,z,t) = u_s(r,\theta,z) + U(r,\theta,z,t)$$

VIII.3.3.1 LONG SOLID CYLINDER long solid cylinder with angular symmetry: $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial \theta} = 0$



BASIC CASE Homogeneous Equation and Boundary Conditions

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad u(r,t): \quad r \in [0, r_1), \qquad t > 0$$

Initial condition: $u(r,0) = u_0(r)$

Boundary conditions: $u(0,t) < \infty \qquad t > 0 \qquad$ *bounded*

$[u]_{r=r_1} = 0 \qquad t > 0 \qquad$ (*I, II or III^rd^ kind*)

1) Separation of variables: $u(r,t) = R(r)T(t) \qquad u(r,t)$ *bounded* $\Rightarrow R(0) < \infty$

$[R(r_1)]T(t) = 0 \Rightarrow [R]_{r=r_1} = 0$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = \frac{1}{\alpha}\frac{T'}{T} = \mu$$ separate variables in the equation

2) Sturm-Liouville Problem: $\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = \mu$

See VII.12, p.509

$r^2 R'' + rR + \left[\lambda^2 r^2 - 0^2\right] R = 0$ *Bessel Equation of 0^{th} order*

$\mu_n = -\lambda_n^2 \qquad n = 1, 2, \ldots$ are positive roots of

$J_0(\lambda_n r_1) = 0$ characteristic equation

$R_n(r) = J_0(\lambda_n r)$ Eigenfunctions

$\left\| R_n(r) \right\|_p^2 = \int_0^{r_1} J_0^2(\lambda_n r)\, r\, dr = \left(r_1^2/2\right) J_1^2(\lambda_n r_1)$ norm

$p(r) = r$ weight function:

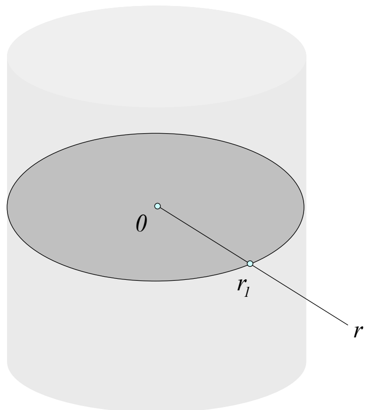
3) Equation for T : $T' - \alpha\mu T = 0$

$T' + \alpha\lambda_n^2 T = 0 \qquad \Rightarrow \qquad T_n(t) = e^{-\alpha\lambda_n^2 t}$

4) Solution: $$u(r,t) = \sum_{n=1}^{\infty} a_n R_n T_n = \sum_{n=1}^{\infty} a_n J_0(\lambda_n r) e^{-\alpha\lambda_n^2 t}$$, where

Initial condition $$u(r,0) = u_0(r) = \sum_{n=1}^{\infty} a_n J_0(\lambda_n r) \quad \Rightarrow \quad a_n = \frac{\int_0^{r_1} u_0(r) J_0(\lambda_n r)\, r dr}{\int_0^{r_1} J_0^2(\lambda_n r)\, r dr}$$

GENERAL CASE: Non-Homogeneous Equation, Non-Homogeneous Boundary Conditions



$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+F(r)=\frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad u(x,t): \quad r\in(0,r_1),\ t>0$$

Initial condition: $u(r,0)=u_0(r)$

Boundary conditions: $u(r,t)<\infty \quad t>0 \quad$ *bounded*

$[u]_{r=r_1}=f_1 \quad t>0 \quad$ (*I, II or IIIrd kind*)

I Steady State Solution

Time-independent solution $u_s(r)$

Substitution of a time-independent function into the heat equation leads to the following ordinary differential equation:

$$\frac{\partial^2 u_s}{\partial r^2}+\frac{1}{r}\frac{\partial u_s}{\partial r}+F(r)=0 \qquad u_s(r),\quad r\in(0,r_1)$$

subject to the boundary conditions of the same kind as for PDE:

$[u_s]_{x=0}$, $\quad t>0 \quad$ bounded

$[u_s]_{r=r_1}=f_1$, $\quad t>0 \quad$ (*I, II or IIIrd kind*)

General solution of ODE:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_s}{\partial r}\right)+F(r)=0$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial u_s}{\partial r}\right)=-rF(r)$$

$$\frac{\partial u_s}{\partial r}=\frac{1}{r}\int\left[-rF(r)\right]dr+\frac{c_1}{r}$$

$$u_s(r)=\int\left\{\frac{1}{r}\int\left[-r\,F(r)\right]dr\right\}dr+c_1\ln r+c_2$$

For bounded solution, it is necessarily $c_1=0$, therefore the general steady state solution in circular domain is

$$u_s(r)=\int\left\{\frac{1}{r}\int\left[-rF(r)\right]dr\right\}dr+c_2$$

Solutions of BVPs for circular domain with uniform heat generation are provided by the Table.

II Transient Solution:

Define the transient solution by equation:

$$U(r,t)=u(r,t)-u_s(r)$$

then solution of the original problem is a sum of transient solution and steady state solution:

$$u(r,t)=U(r,t)+u_s(r)$$

Substitute it into the Heat Equation:

$$\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}+\frac{\partial^2 u_s}{\partial r^2}+\frac{1}{r}\frac{\partial u_s}{\partial r}+F(r)=\frac{1}{\alpha}\frac{\partial U}{\partial t}$$

Since $\frac{\partial^2 u_s}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+F(x)=0$, it yields

$$\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}=\frac{1}{\alpha}\frac{\partial U}{\partial t}$$

We obtained the equation for the new unknown function $U(r,t)$ which has homogeneous boundary condition:

$r=r_1 \qquad [U]_{r=r_1}=[u]_{r=r_1}-[u_s]_{r=r_1}=f_1-f_1=0$

As a result, we reduced the non-homogeneous problem to a homogeneous equation for $U(r,t)$ with homogeneous boundary conditions. Initial condition for function $U(r,t)$:

$U(r,0)=u(r,0)-u_s(r)=u_0(r)-u_s(r)$

Solution for *U(r,t)*

We consider the following BASIC initial boundary value problem:

$$\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}=\frac{1}{\alpha}\frac{\partial U}{\partial t} \qquad U(r,t),\quad r\in(0,r_1),\ t>0$$

initial condition: $\qquad U(r,0)=u_0(r)-u_s(r)$

boundary conditions: $\qquad U(0,t)<\infty \qquad t>0$

$\qquad [U]_{r=r_1}=0 \qquad t>0$

We already know a solution of this basic problem obtained by separation of variables:

$$U(x,t)=\sum_{n=1}^{\infty}a_nR_nT_n=\sum_{n=1}^{\infty}a_nJ_0(\lambda_n r)e^{-\alpha\lambda_n^2 t}$$

where coefficients a_n are the Fourier coefficients determined by the corresponding initial condition for the function $U(x,t)$:

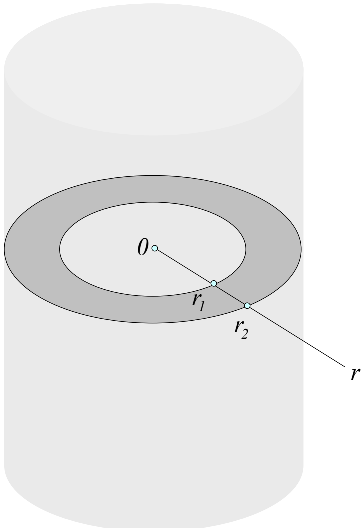
$$a_n=\frac{\int_0^{r_1}\left[u_0(r)-u_s(r)\right]R_n(r)rdr}{\int_0^{r_1}R_n^2(r)rdr}=\frac{\int_0^{r_1}\left[u_0(r)-u_s(r)\right]J_0(\lambda_n r)rdr}{\int_0^{r_1}J_0^2(\lambda_n r)rdr}$$

<u>III Solution of IBVP:</u>

Solution of the original IBVP is a sum of steady state solution and transient solution:

$$\boxed{u(r,t)=u_s(r)+U(r,t)}$$

VIII.3.3.2 HOLLOW CYLINDER BASIC CASE: Homogeneous Equation and Boundary Conditions



$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}=\frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad u(x,t): \quad r\in(0,r_1),\ t>0$$

Initial condition: $u(r,0)=u_0(r)$

Boundary conditions:
$[u]_{r=r_1}=0 \quad t>0 \quad$ (*I, II or IIIrd kind*)
$[u]_{r=r_2}=0 \quad t>0 \quad$ (*I, II or IIIrd kind*)

1) Separation of variables: $u(r,t)=R(x)T(t)$

$$[u]_{r=r_1}=[R]_{r=r_1}T(t)=0 \quad\Rightarrow\quad [R]_{r=r_1}=0$$
$$[u]_{r=r_2}=[R]_{r=r_2}T(t)=0 \quad\Rightarrow\quad [R]_{r=r_2}=0$$

$$\frac{R''}{R}+\frac{1}{r}\frac{R'}{R}=\frac{1}{\alpha}\frac{T'}{T}=\mu$$

2) Sturm-Liouville Problem: $\frac{R''}{R}+\frac{1}{r}\frac{R'}{R}=\mu \qquad$ ($\mu=-\lambda^2$ SLP)

$$r^2R''+rR+\left[\lambda^2r^2-0^2\right]R=0 \qquad \textit{Bessel Equation of 0 order}$$

Eigenvalues: $\mu_n=-\lambda_n^2 \quad n=1,2,\ldots$
λ_n are roots of characteristic eqn

Eigenfunctions: $R_n(r)=c_{1,n}J_0(\lambda_n r)+c_{2,n}Y_0(\lambda_n r)$

3) Equation for T: $T'-\alpha\mu T=0$

$$T'+\alpha\lambda_n^2T=0 \quad\Rightarrow\quad T_n(t)=e^{-\alpha\lambda_n^2 t}$$

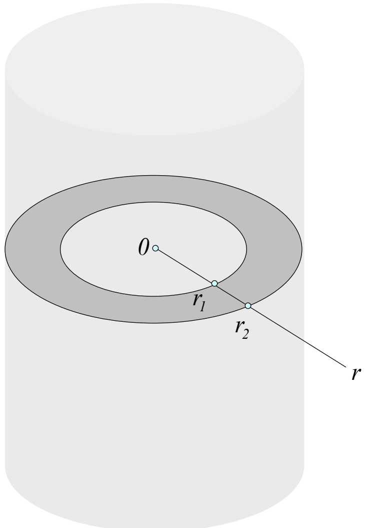
4) Solution:

$$u(r,t)=\sum_{n=1}^{\infty}a_nR_nT_n \qquad =\sum_{n=1}^{\infty}a_nR_n(r)e^{-\alpha\lambda_n^2 t}$$

Initial condition:

$$u(r,0)=u_0(r)=\sum_{n=1}^{\infty}a_nR_n \quad\Rightarrow\quad a_n=\frac{\int_0^{r_1}u_0(r)R_n(r)rdr}{\int_0^{r_1}R_n^2(r)rdr}$$

GENERAL CASE: Non-Homogeneous Equation, Non-Homogeneous Boundary Conditions



$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+F(r)=\frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad u(x,t): \quad r\in(0,r_1)\,,\ t>0$$

Initial condition: $u(r,0)=u_0(r)$

Boundary conditions: $[u]_{r=r_1}=f_1 \quad t>0 \qquad$ (*I, II or IIIrd kind*)

$[u]_{r=r_2}=f_2 \quad t>0 \qquad$ (*I, II or IIIrd kind*)

I Steady State Solution

Time-independent solution $u_s(r)$

Substitution of a time-independent function into the heat equation leads to the following ordinary differential equation:

$$\frac{\partial^2 u_s}{\partial r^2}+\frac{1}{r}\frac{\partial u_s}{\partial r}+F(r)=0 \qquad u_s(r)\,,\quad r\in(0,r_1)$$

subject to the boundary conditions of the same kind as for PDE:

$$[u_s]_{r=r_1}=f_1\,, \qquad t>0 \qquad (\textit{I, II or IIIrd kind})$$

$$[u_s]_{r=r_2}=f_2\,, \qquad t>0 \qquad (\textit{I, II or IIIrd kind})$$

General solution of ODE:

$$u_s(r)=\int\left\{\frac{1}{r}\int\left[-rF(r)\right]dr\right\}dr+c_1\ln r+c_2$$

Coefficients c_1,c_2 have to be determined from boundary conditions.

II Transient Solution:

Define the transient solution by equation:

$$U(r,t)=u(r,t)-u_s(r)\,, \qquad u(r,t)=U(r,t)+u_s(r)$$

Substitution into the Heat Equation yields an equation for transient solution:

$$\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}=\frac{1}{\alpha}\frac{\partial U}{\partial t}$$

for the new unknown function $U(r,t)$ which has two homogeneous boundary conditions:

$$r=r_1 \qquad [U]_{r=r_1}=[u]_{r=r_1}-[u_s]_{r=r_1}=f_1-f_1=0$$

$$r=r_2 \qquad [U]_{r=r_2}=[u]_{r=r_2}-[u_s]_{r=r_2}=f_2-f_2=0$$

and initial condition:

$$U(r,0)=u(r,0)-u_s(r)=u_0(r)-u_s(r)$$

Solution for *U(r,t)*

We consider the following BASIC initial boundary value problem:

$$\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}=\frac{1}{\alpha}\frac{\partial U}{\partial t} \qquad U\left(r,t\right),\quad r\in\left(0,r_1\right),\ t>0$$

initial condition: $U\left(r,0\right)=u_0\left(r\right)-u_s\left(r\right)$

boundary conditions:

$$\left[U\right]_{r=r_1}=0 \qquad t>0$$

$$\left[U\right]_{r=r_2}=0 \qquad t>0$$

We already know a solution of this basic problem obtained by separation of variables:

$$U\left(x,t\right)\quad=\sum_{n=1}^{\infty}a_nR_nT_n\quad=\sum_{n=1}^{\infty}a_nR_n\left(r\right)e^{-\alpha\lambda_n^2 t}$$

where coefficients a_n are the Fourier coefficients determined by the corresponding initial condition for the function $U\left(x,t\right)$:

$$a_n=\frac{\int_0^{r_1}\left[u_0\left(r\right)-u_s\left(r\right)\right]R_n\left(r\right)rdr}{\int_0^{r_1}R_n^2\left(r\right)rdr}$$

<u>III Solution of IBVP:</u>

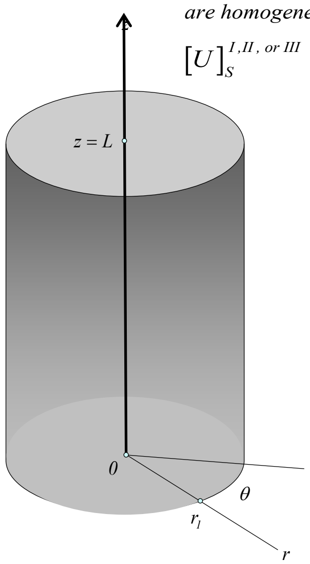
Solution of the original IBVP is a sum of steady state solution and transient solution:

$$u\left(r,t\right)\quad=u_s\left(r\right)+U\left(r,t\right)$$

VIII.3.3.5 BASIC IBVP FOR HEAT EQUATION IN FINITE SOLID CYLINDER 3-D

all boundary conditions are homogeneous

$$\left[U\right]_S^{I,II,\text{ or } III} = 0$$



initial condition:

$$U\left(r,\theta,z,t=0\right) = U_0\left(r,\theta,z\right)$$

The ***Basic IBVP for the 3-D Heat Equation (transient solution)***:

$$\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}+\frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}+\frac{\partial^2 U}{\partial z^2}=\frac{1}{\alpha}\frac{\partial U}{\partial t}$$

$$\left(r,\theta,z\right)\in\left[0,r_1\right)\times\left[0,2\pi\right]\times\left(0,L\right)\ ,\quad t>0$$

$$\left(r,\theta,z\right)\in\left[0,r_1\right)\times\left[-\pi,\pi\right]\times\left(0,L\right)$$

Separation of variables:

$$U\left(r,\theta,z,t\right)=\Phi\left(r,\theta,z\right)T\left(t\right)$$

Separated equation:

$$\frac{\dfrac{\partial^2 \Phi}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial \Phi}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 \Phi}{\partial \theta^2}+\dfrac{\partial^2 \Phi}{\partial z^2}}{\Phi}=\frac{1}{\alpha}\frac{T'}{T}=\beta$$

Separated equation yields the Helmholtz Equation:

Helmholtz Equation

$$\boxed{\frac{\nabla^2\Phi}{\Phi}=\beta} \qquad \boxed{\nabla^2\Phi=\beta\Phi}$$

The solution of the Helmholtz Equation subject to boundary conditions can be obtained by the eigenfunction expansion method.

Assume

$$\Phi\left(r,\theta,z\right)=R\left(r\right)\Theta\left(\theta\right)Z\left(z\right)$$

Substitute into the Helmholtz Equation

$$\frac{R''}{R}+\frac{1}{r}\frac{R'}{R}+\frac{1}{r^2}\frac{\Theta''}{\Theta}+\frac{Z''}{Z}=\beta$$

boundary conditions Separation of variables in the boundary conditions yields:

$$r=r_1 \qquad \left[\Phi\right]_{r=r_1}=\left[R\left(r_1\right)\right]\Theta\left(\theta\right)Z\left(z\right)=0 \quad\Rightarrow\quad \left[R\right]_{r=r_1}=0$$

$$z=0 \qquad \left[\Phi\right]_{z=0}=R\left(r\right)\Theta\left(\theta\right)\left[Z\left(0\right)\right]=0 \quad\Rightarrow\quad \left[Z\right]_{z=0}=0$$

$$z=L \qquad \left[\Phi\right]_{z=K}=R\left(r\right)\Theta\left(\theta\right)\left[Z\left(K\right)\right]=0 \quad\Rightarrow\quad \left[Z\right]_{z=L}=0$$

From physical consideration, we need

bounded solution

$$r=0 \qquad \left[\Phi\right]_{r=0}=\left[R\left(0\right)\right]\Theta\left(\theta\right)Z\left(z\right)<\infty \quad\Rightarrow\quad \left[R\right]_{r=0}<\infty$$

2π -periodic solution

$$\Phi\left(r,\theta+2\pi,z\right)=\Phi\left(r,\theta+2\pi,z\right) \quad\Rightarrow\quad \Theta\left(\theta+2\pi\right)=\Theta\left(\theta\right)$$

Separate variables $$\frac{\Theta''}{\Theta} = r^2\beta - r^2\frac{R''}{R} - r\frac{R'}{R} - r^2\frac{Z''}{Z} = \eta$$

1st equation

$$\Theta'' - \eta\Theta = 0$$

that is the SLP without boundary conditions, with condition of periodicity $\Theta(\theta + 2\pi) = \Theta(\theta)$ (see also the section VIII.3.6).

It can be considered in the interval $-\pi \le \theta < \pi$ with the condition $\Theta(-\pi) = \Theta(\pi)$

The case $\eta = 0$ yields the linear solution
$\Theta_0 = c_1\theta + c_2$
The only periodic linear function is a constant function, therefore,
$\Theta_0 = 1$
can be taken as an eigenfunction corresponding to $\eta_0 = 0$.

For positive eigenvalues, the separation constant has to be $\eta = -\mu^2$, then the general solution is

$\Theta_0 = c_1 \cos \mu\theta + c_2 \sin \mu\theta$

A function with a period $T = \frac{2\pi}{n}$ is also a 2π -periodic. Therefore, for 2π –periodic solution, the frequency μ can be any positive integer

$\mu = \frac{2\pi}{T} = \frac{2\pi}{2\pi}n = n$.

So, for $\eta_n = -n^2$, the corresponding eigenfunctions are

$\Theta_n = c_1 \cos n\theta + c_2 \sin n\theta$

That is consistent with the standard Fourier series over symmetric 2π -interval $(-\pi,\ \pi)$, which is based on the complete set of mutually orthogonal functions:

$\{1,\ \cos(n\theta),\ \sin(n\theta)\}$

Therefore, solution of the first equation can be summarized as:

$\eta_0 = 0$	$\Theta_0(\theta) = 1$	
$\eta_n = -n^2$	$\Theta_n(\theta) = a_n \cos(n\theta) + b_n \sin(n\theta)$	$n = 1, 2, \ldots$

2nd equation

$$r^2\beta - r^2\frac{R''}{R} - r\frac{R'}{R} - r^2\frac{Z''}{Z} = -n^2 \qquad n = 0, 1, 2, \ldots$$

$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} - \frac{n^2}{r^2} = \beta - \frac{Z''}{Z} = \mu$ separate variables

$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} - \frac{n^2}{r^2} = \mu$ consider equation for R

$$r^2 R'' + rR' - n^2 R - \mu r^2 R = 0$$

That is the Sturm-Liouville problem for Bessel Equation of order n

$$r^2 R'' + rR' + \left[(-\mu) r^2 - n^2 \right] R = 0 \qquad (rR')' + \left[\frac{-n^2}{r} + (-\mu) r \right] R = 0$$

$$\mu = -\lambda^2$$

$$r^2 R'' + rR' + \left[\lambda^2 r^2 - n^2 \right] R = 0$$ (see sectionVII.12, p.507)

$$[R]_{r=0} < \infty$$

$$[R]_{r=r_1} = 0$$

$$R_n(r) = c_{1,n} J_n(\lambda r) + c_{2,n} Y_n(\lambda r)$$ *general solution*

$$[R]_{r=0} < \infty \quad \Rightarrow \quad c_{2,n} = 0$$

$$R_n(r) = c_{1,n} J_n(\lambda r)$$

$$[R]_{r=r_1} = 0 \quad \Rightarrow \quad [J_n(\lambda r_1)] = 0 \quad \Rightarrow \quad \lambda_{mn} \qquad n = 0,1,2,\ldots$$
$$m = (0),1,2,\ldots$$

$$R_n(r) = J_n(\lambda r)$$

$$[J_n(\lambda r_1)] = 0 \quad \Rightarrow \quad \lambda_{mn} \qquad n = 0,1,2,\ldots$$
$$m = (0),1,2,\ldots$$

n comes from the order of the Bessel functions $J_n(\lambda r)$.
Eigenvalues λ_{mn} should be found for each $n = 0,1,2,\ldots$

3rd equation

$$\beta - \frac{Z''}{Z} = -\lambda_{mn}^2$$

$$\frac{Z''}{Z} = \lambda_{mn}^2 + \beta = \gamma$$ combine constants to a single parameter γ

$$Z'' - \gamma Z = 0$$

$$[Z]_{z=0} = 0 \quad \overset{SLP}{\Longrightarrow} \quad \gamma = -\omega_k^2 \qquad k = (0),1,2,\ldots$$

$$[Z]_{z=K} = 0 \qquad Z_k(z)$$ eigenfunctions

Then the second part of the last equation becomes

$$\lambda_{mn}^2 + \beta = -\omega_k^2$$

and the constant of separation is

$$\beta_{mnk} = -\left(\lambda_{mn}^2 + \omega_k^2 \right)$$

Solution for $T(t)$

$$T_{nmk}(t) = e^{-\alpha\left(\lambda_{mn}^2+\omega_k^2\right)t}$$

The solution of the Basic IBVP for the Heat Equation is

$$U(r,\theta,z,t) = \overset{n=0}{\sum_m\sum_k A_{0mk} J_0(\lambda_{0m}r) Z_k(z) e^{-\alpha\left(\lambda_{0m}^2+\omega_k^2\right)t}} + \sum_{n=1}\sum_m\sum_k\left[A_{nmk}\cos(n\theta)+B_{nmk}\sin(n\theta)\right]J_n(\lambda_{nm}r)Z_k(z)e^{-\alpha\left(\lambda_{nm}^2+\omega_k^2\right)t}$$

The coefficients in this solution should be found to satisfy the initial condition:

$$U(r,\theta,z,0) = U_0(r,\theta,z)$$

$$U_0(r,\theta,z) = \sum_m\sum_k A_{0mk}J_0(\lambda_{0m}r)Z_k(z) + \sum_{n=1}\left[\sum_m\sum_k A_{nmk}Z_k(z)J_n(\lambda_{nm}r)\right]\cos(n\theta) + \sum_{n=1}\left[\sum_m\sum_k B_{nmk}Z_k(z)J_n(\lambda_{nm}r)\right]\sin(n\theta)$$

The following cascade of expansions over the eigenfunctions yields equations for calculation of coefficients. First, they are calculated as the coefficients of the standard Fourier series over interval $(-\pi,\pi)$:

$$\sum_m\left[\sum_k A_{0mk}Z_k(z)\right]J_0(\lambda_{0m}r) = \frac{1}{2\pi}\int_{-\pi}^{+\pi}U_0(r,\theta,z)\,d\theta$$

$$\sum_m\left[\sum_k A_{nmk}Z_k(z)\right]J_n(\lambda_{nm}r) = \frac{1}{\pi}\int_{-\pi}^{+\pi}U_0(r,\theta,z)\cos(n\theta)\,d\theta$$

$$\sum_m\left[\sum_k B_{nmk}Z_k(z)\right]J_n(\lambda_{nm}r) = \frac{1}{\pi}\int_{-\pi}^{+\pi}U_0(r,\theta,z)\sin(n\theta)\,d\theta$$

Second, as the coefficients of expansion into Fourier-Bessel series:

$$\sum_k A_{0mk}Z_k(z) = \frac{1}{2\pi\left\|J_0(\lambda_{0m})\right\|^2}\int_{-\pi}^{+\pi}\int_0^{r_1}U_0(r,\theta,z)\,J_0(\lambda_{0m}r)\;rdr\,d\theta$$

$$\sum_k A_{nmk}Z_k(z) = \frac{1}{\pi\left\|J_n(\lambda_{nm})\right\|^2}\int_{-\pi}^{+\pi}\int_0^{r_1}U_0(r,\theta,z)\,J_n(\lambda_{nm}r)\;\cos(n\theta)rdr\,d\theta$$

$$\sum_k B_{nmk}Z_k(z) = \frac{1}{\pi\left\|J_n(\lambda_n)\right\|^2}\int_{-\pi}^{+\pi}\int_0^{r_1}U_0(r,\theta,z)\,J_n(\lambda_{nm}r)\;\sin(n\theta)rdr\,d\theta$$

and, finally, by expansion into Generalized Fourier series, the coefficients for solution of the Basic IBVP are defined

$$A_{0mk} = \frac{1}{2\pi\left\|J_0(\lambda_{0m}r)\right\|^2\left\|Z_k(z)\right\|^2}\int_0^L\int_{-\pi}^{+\pi}\int_0^{r_1}U_0(r,\theta,z)\,J_0(\lambda_{0m}r)\;Z_k(z)rdr\,d\theta\,dz$$

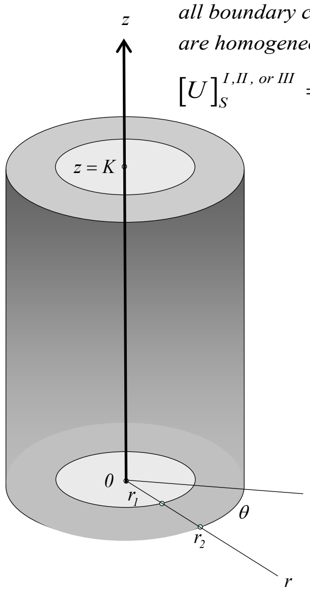
$$A_{nmk} = \frac{1}{\pi\left\|J_n(\lambda_{nm}r)\right\|^2\left\|Z_k(z)\right\|^2}\int_0^L\int_{-\pi}^{+\pi}\int_0^{r_1}U_0(r,\theta,z)\,J_n(\lambda_{nm}r)\;Z_k(z)\;\cos(n\theta)rdr\,d\theta\,dz$$

$$B_{nmk} = \frac{1}{\pi\left\|J_n(\lambda_{nm}r)\right\|^2\left\|Z_k(z)\right\|^2}\int_0^L\int_{-\pi}^{+\pi}\int_0^{r_1}U_0(r,\theta,z)\,J_n(\lambda_{nm}r)\;Z_k(z)\;\sin(n\theta)rdr\,d\theta\,dz$$

VIII.3.3.6 BASIC IBVP FOR HEAT EQUATION IN FINITE HOLLOW CYLINDER 3-D

all boundary conditions are homogeneous

$$\left[U\right]_S^{I,II,\text{ or }III} = 0$$



initial condition:

$$U\left(r,\theta,z,t=0\right) = U_0\left(r,\theta,z\right)$$

Consider the Helmholtz equation which appears in the separation of variables in the ***Basic IBVP for the 3-D Heat Equation***:

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2} = \frac{1}{\alpha}\frac{\partial U}{\partial t}$$

$$\left(r,\theta,z\right) \in \left(r_1,r_2\right)\times\left[0,2\pi\right]\times\left(0,L\right) \;,\quad t>0$$

$$\left(r,\theta,z\right) \in \left(r_1,r_2\right)\times\left[-\pi,\pi\right]\times\left(0,L\right)$$

Separation of variables:

$$U\left(r,\theta,z,t\right) = \Phi\left(r,\theta,z\right)T\left(t\right)$$

Separated equation:

$$\frac{\dfrac{\partial^2 \Phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial \Phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \Phi}{\partial \theta^2} + \dfrac{\partial^2 \Phi}{\partial z^2}}{\Phi} = \frac{1}{\alpha}\frac{T'}{T} = \beta$$

Separated equation yields the Helmholtz Equation:

Helmholtz Equation

$$\boxed{\frac{\nabla^2 \Phi}{\Phi} = \beta} \qquad \boxed{\nabla^2 \Phi = \beta\Phi}$$

The solution of the Helmholtz Equation subject to boundary conditions can be obtained by the eigenfunction expansion method.

Assume

$$\Phi\left(r,\theta,z\right) = R\left(r\right)\Theta\left(\theta\right)Z\left(z\right)$$

Substitute into the Helmholtz Equation

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{Z''}{Z} = \beta$$

boundary conditions

Separation of variables in the boundary conditions yields:

$r = r_1$ $\quad \left[\Phi\right]_{r=r_1} = \left[R\left(r_1\right)\right]\Theta\left(\theta\right)Z\left(z\right) = 0 \quad \Rightarrow \quad \left[R\right]_{r=r_1} = 0$

$r = r_2$ $\quad \left[\Phi\right]_{r=r_2} = \left[R\left(r_2\right)\right]\Theta\left(\theta\right)Z\left(z\right) = 0 \quad \Rightarrow \quad \left[R\right]_{r=r_2} = 0$

$z = 0$ $\quad \left[\Phi\right]_{z=0} = R\left(r\right)\Theta\left(\theta\right)\left[Z\left(0\right)\right] = 0 \quad \Rightarrow \quad \left[Z\right]_{z=0} = 0$

$z = L$ $\quad \left[\Phi\right]_{z=K} = R\left(r\right)\Theta\left(\theta\right)\left[Z\left(K\right)\right] = 0 \quad \Rightarrow \quad \left[Z\right]_{z=K} = 0$

From physical consideration, we need

2π -periodic solution $\quad \Phi\left(r,\theta+2\pi,z\right) = \Phi\left(r,\theta+2\pi,z\right) \quad \Rightarrow \quad \Theta\left(\theta+2\pi\right) = \Theta\left(\theta\right)$

Separate variables

$$\frac{\Theta''}{\Theta} = r^2\beta - r^2\frac{R''}{R} - r\frac{R'}{R} - r^2\frac{Z''}{Z} = \eta$$

1st equation

$$\Theta'' - \eta\Theta = 0$$

that is the SLP without boundary conditions, with condition of periodicity $\Theta(\theta + 2\pi) = \Theta(\theta)$ (see also the section VIII.3.6).

It can be considered in the interval $-\pi \le \theta < \pi$ with the condition $\Theta(-\pi) = \Theta(\pi)$

The case $\eta = 0$ yields the linear solution

$\Theta_0 = c_1\theta + c_2$

The only periodic linear function is a constant function, therefore,

$\Theta_0 = 1$

can be taken as an eigenfunction corresponding to $\eta_0 = 0$.

For positive eigenvalues, the separation constant has to be $\eta = -\mu^2$, then the general solution is

$\Theta_0 = c_1 \cos\mu\theta + c_2 \sin\mu\theta$

A function with a period $T = \dfrac{2\pi}{n}$ is also a 2π -periodic. Therefore, for 2π –periodic solution, the frequency μ can be any positive integer

$\mu = \dfrac{2\pi}{T} = \dfrac{2\pi}{2\pi}n = n$.

So, for $\eta_n = -n^2$, the corresponding eigenfunctions are

$\Theta_n = c_1 \cos n\theta + c_2 \sin n\theta$

That is consistent with the standard Fourier series over symmetric 2π -interval $(-\pi,\ \pi)$, which is based on the complete set of mutually orthogonal functions:

$\{1,\ \cos(n\theta),\ \sin(n\theta)\}$

Therefore, solution of the first equation can be summarized as:

$$\eta_0 = 0 \qquad \Theta_0(\theta) = 1$$

$$\eta_n = -n^2 \qquad \Theta_n(\theta) = a_n \cos(n\theta) + b_n \sin(n\theta) \qquad n = 1,2,...$$

2nd equation

$$r^2\beta - r^2\frac{R''}{R} - r\frac{R'}{R} - r^2\frac{Z''}{Z} = -n^2 \qquad n = 0,1,2,...$$

$\dfrac{R''}{R} + \dfrac{1}{r}\dfrac{R'}{R} - \dfrac{n^2}{r^2} = \beta - \dfrac{Z''}{Z} = \mu$ separate variables

$\dfrac{R''}{R} + \dfrac{1}{r}\dfrac{R'}{R} - \dfrac{n^2}{r^2} = \mu$ consider equation for R

$$r^2 R'' + rR' - n^2 R - \mu r^2 R = 0$$

That is the Sturm-Liouville problem for Bessel Equation of order n

$$r^2 R'' + rR' + \left[(-\mu) r^2 - n^2 \right] R = 0 \qquad (rR')' + \left[\frac{-n^2}{r} + (-\mu) r \right] R = 0$$

$$\mu = -\lambda^2$$

$$r^2 R'' + rR' + \left[\lambda^2 r^2 - n^2 \right] R = 0$$ (see sectionVII.12, p.515)

$$[R]_{r=r_1} = 0$$

$$[R]_{r=r_2} = 0$$

$$R_n(r) = c_{1,n} J_n(\lambda r) + c_{1,n} Y_n(\lambda r)$$ *general solution*

See solution of the Sturm-Liouville problem for the Bessel equation in the annular domain (Section VII.12, p.515):
For each $n = 0,1,2,...$, there infinitely many eigenvalues λ_{nm} and corresponding eigenfunctions (orthogonal w.r.t weight $p(r) = r$):

$$R_{nm}(r) = c_{1,n} J_n(\lambda_{nm} r) + c_{1,n} Y_n(\lambda_{nm} r)$$

$$[characteristic\ eqn] = 0 \qquad \Rightarrow \qquad \lambda_{mn} \qquad \begin{array}{l} n = 0,1,2,... \\ m = (0),1,2,... \end{array}$$

n comes from the order of the Bessel functions $J_n(\lambda r)$ and $Y_n(\lambda r)$.
Eigenvalues λ_{mn} should be found for each $n = 0,1,2,...$
The square of the norm of eigenfunctions is denoted as $\|R_{nm}(r)\|^2$

3rd equation

$$\beta - \frac{Z''}{Z} = -\lambda_{mn}^2$$

$$\frac{Z''}{Z} = \lambda_{mn}^2 + \beta = \gamma$$ combine constants to a single parameter γ

$$Z'' - \gamma Z = 0$$

$$[Z]_{z=0} = 0 \qquad \overset{SLP}{\Longrightarrow} \qquad \gamma = -\omega_k^2 \qquad k = (0),1,2,...$$

$$[Z]_{z=K} = 0 \qquad \qquad Z_k(z)$$ eigenfunctions

Then the second part of the last equation becomes

$$\lambda_{mn}^2 + \beta = -\omega_k^2$$

and the constant of separation is

$$\beta_{mnk} = -\left(\lambda_{mn}^2 + \omega_k^2 \right)$$

Solution for $T(t)$

$$T_{nmk}(t) = e^{-\alpha\left(\lambda_{mn}^2+\omega_k^2\right)t}$$

The solution of the Basic IBVP for the Heat Equation is:

$$R_{0m}(r) = J_0\left(\lambda_{0mr}\right)$$

$$U(r,\theta,z,t) = \sum_m \sum_k A_{0mk} R_{0m}(r) Z_k(z) e^{-\alpha\left(\lambda_{0m}^2+\omega_k^2\right)t} + \sum_{n=1} \sum_m \sum_k \left[A_{nmk} \cos(n\theta) + B_{nmk} \sin(n\theta) \right] R_{nm}(\lambda_{nm} r) Z_k(z) e^{-\alpha\left(\lambda_{nm}^2+\omega_k^2\right)t}$$

The coefficients in this solution should be found to satisfy the initial condition:

$$U(r,\theta,z,0) = U_0(r,\theta,z)$$

$$U_0(r,\theta,z) = \sum_m \sum_k A_{0mk} R_{0m}(\lambda_{0m} r) Z_k(z) + \sum_{n=1} \left[\sum_m \sum_k A_{nmk} Z_k(z) R_{nm}(\lambda_{nm} r) \right] \cos(n\theta) + \sum_{n=1} \left[\sum_m \sum_k B_{nmk} Z_k(z) R_{nm}(\lambda_{nm} r) \right] \sin(n\theta)$$

The following cascade of expansions over the eigenfunctions yields equations for calculation of coefficients.
First, they are calculated as the coefficients of the standard Fourier series over interval $(-\pi,\pi)$:

$$\sum_m \left[\sum_k A_{0mk} Z_k(z) \right] R_{0m}(\lambda_{0m} r) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} U_0(r,\theta,z)\, d\theta$$

$$\sum_m \left[\sum_k A_{nmk} Z_k(z) \right] R_{nm}(\lambda_{nm} r) = \frac{1}{\pi} \int_{-\pi}^{+\pi} U_0(r,\theta,z) \cos(n\theta) d\theta$$

$$\sum_m \left[\sum_k B_{nmk} Z_k(z) \right] R_{nm}(\lambda_{nm} r) = \frac{1}{\pi} \int_{-\pi}^{+\pi} U_0(r,\theta,z) \sin(n\theta) d\theta$$

Second, as the coefficients of expansion into Fourier-Bessel series:

$$\sum_k A_{0mk} Z_k(z) = \frac{1}{2\pi \left\| R_{0m}(\lambda_{0m}) \right\|^2} \int_{-\pi}^{+\pi} \int_{r_1}^{r_2} U_0(r,\theta,z)\, R_{0m}(\lambda_{0m} r)\, r dr\, d\theta$$

$$\sum_k A_{nmk} Z_k(z) = \frac{1}{\pi \left\| R_{nm}(\lambda_{nm}) \right\|^2} \int_{-\pi}^{+\pi} \int_{r_1}^{r_2} U_0(r,\theta,z)\, R_{nm}(\lambda_{nm} r)\, \cos(n\theta) r\, dr\, d\theta$$

$$\sum_k B_{nmk} Z_k(z) = \frac{1}{\pi \left\| R_{nm}(\lambda_{nm}) \right\|^2} \int_{-\pi}^{+\pi} \int_{r_1}^{r_2} U_0(r,\theta,z)\, R_{nm}(\lambda_{nm} r)\, \sin(n\theta) r\, dr\, d\theta$$

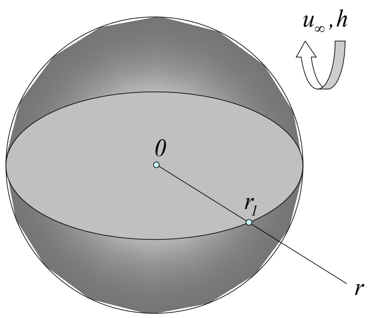
and, finally, by expansion into Generalized Fourier series, the coefficients for solution of the Basic IBVP are defined

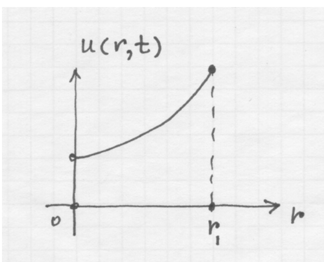
$$A_{0mk} = \frac{1}{2\pi \left\| R_{0m}(\lambda_{0m} r) \right\|^2 \left\| Z_k(z) \right\|^2} \int_0^L \int_{-\pi}^{+\pi} \int_{r_1}^{r_2} U_0(r,\theta,z)\, R_{0m}(\lambda_{0m} r)\, Z_k(z)\, r\, dr\, d\theta\, dz$$

$$A_{nmk} = \frac{1}{\pi \left\| R_{nm}(\lambda_{nm} r) \right\|^2 \left\| Z_k(z) \right\|^2} \int_0^L \int_{-\pi}^{+\pi} \int_{r_1}^{r_2} U_0(r,\theta,z)\, R_{nm}(\lambda_{nm} r)\, Z_k(z)\, \cos(n\theta) r\, dr\, d\theta\, dz$$

$$B_{nmk} = \frac{1}{\pi \left\| R_{nm}(\lambda_{nm} r) \right\|^2 \left\| Z_k(z) \right\|^2} \int_0^L \int_{-\pi}^{+\pi} \int_{r_1}^{r_2} U_0(r,\theta,z)\, R_{nm}(\lambda_{nm} r)\, Z_k(z)\, \sin(n\theta) r\, dr\, d\theta\, dz$$

VIII.3.4 SOLID SPHERE





Consider **heat conduction** in the sphere with angular symmetry:

$$\frac{\partial u}{\partial \phi} = \frac{\partial u}{\partial \theta} = 0$$

he non-stationary temperature field $u(r,t)$ depends only on the radial variable *r* and time variable *t*.

Initial-boundary value problem:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{\dot q(r)}{k}=\frac{1}{\alpha}\frac{\partial u}{\partial t} \qquad r\in[0,r_1) \qquad t>0$$

initial condition:

$$u(r,0)=u_0(r) \qquad r\in[0,r_1]$$

boundary conditions:

$$k\left.\frac{\partial u}{\partial r}\right|_{r=r_1}=h\left(u_\infty-u|_{r=r_1}\right) \qquad t>0$$

$$u|_{r=0}<\infty \qquad t>0$$

where u_∞ is the ambient temperature and h is a convective coefficient. Rewrite the boundary condition in the standard form

$$\left[\frac{\partial u}{\partial r}+\frac{h}{k}u\right]_{r=r_1}=\frac{hu_\infty}{k}$$

1) Superposition of Steady State and Transient Solutions:

$$u(r,t)=u_s(r)+U(r,t)$$

2) Steady State Solution:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_s}{\partial r}\right)+\frac{\dot q(r)}{k}=0$$

boundary conditions:

$$\left[\frac{\partial u_s}{\partial r}+\frac{h}{k}u_s\right]_{r=r_1}=\frac{hu_\infty}{k} \qquad t>0$$

$$u_s|_{r=0}<\infty \qquad t>0$$

General solution:

$$u_s(r) = -\int\left[\frac{1}{r^2}\int\frac{\dot q(r)}{k}r^2dr\right]dr-\frac{c_1}{r}+c_2$$

For the solid sphere (bounded solution at $r=0$):

$$u_s(r) = -\int\left[\frac{1}{r^2}\int\frac{\dot q(r)}{k}r^2dr\right]dr+c_2$$

For uniform heat generation ($\dot q = const$):

$$u_s(r) = \frac{\dot q}{6k}\left(r_1^2-r^2\right)+\frac{\dot q r_1}{3h}+u_\infty$$

3) Transient Solution:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right)=\frac{1}{\alpha}\frac{\partial U}{\partial t} \qquad r\in\left[0,r_1\right) \quad t>0$$

initial condition:

$$U\left(r,0\right)=u_0\left(r\right)-u_s\left(r\right) \qquad r\in\left[0,r_1\right]$$

boundary conditions:

$$\left[\frac{\partial U}{\partial r}+\frac{h}{k}U\right]_{r=r_1}=0 \qquad t>0$$

$$U\Big|_{r=0}<\infty \qquad t>0$$

Reduction to Cartesian coordinates

Introduce the new dependent variable (reduction to Cartesian case):

$$V\left(r,t\right) = r\,U\left(r,t\right)$$

Write $U =\frac{1}{r}V$

Evaluate l.h.s. $\frac{\partial}{\partial r}U =-\frac{1}{r^2}V+\frac{1}{r}\frac{\partial}{\partial r}V$

$$r^2\frac{\partial}{\partial r}U =-V+r\frac{\partial}{\partial r}V$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}U\right) =-\frac{\partial}{\partial r}V+\frac{\partial}{\partial r}V+r\frac{\partial^2}{\partial r^2}V$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}U\right) =\frac{1}{r}\frac{\partial^2}{\partial r^2}V$$

Evaluate r.h.s. $\frac{1}{\alpha}\frac{\partial U}{\partial t} =\frac{1}{\alpha}\frac{1}{r}\frac{\partial V}{\partial t}$

Into equation: $\frac{1}{r}\frac{\partial^2}{\partial r^2}V =\frac{1}{\alpha}\frac{1}{r}\frac{\partial V}{\partial t}$

$$\frac{\partial^2 V}{\partial r^2} =\frac{1}{\alpha}\frac{\partial V}{\partial t}$$

which formally is the 1-d Heat Equation for r in the finite interval $r\in\left[0,r_1\right)$, which requires two boundary conditions.

The first condition at $r=0$ is obtained directly from the equation used for a change of variable:

$$V\Big|_{r=0}=rU\Big|_{r=0}=0 \qquad \textit{Dirichlet}$$

Consider the second boundary condition at $r=r_1$:

$$\left[\frac{\partial U}{\partial r}+\frac{h}{k}U\right]_{r=r_1}=0$$

$$\left[\frac{\partial}{\partial r}\frac{V}{r}+\frac{h}{k}\frac{V}{r}\right]_{r=r_1}=0$$

$$\left[\frac{1}{r}\frac{\partial V}{\partial r}-\frac{V}{r^2}+\frac{h}{k}\frac{V}{r}\right]_{r=r_1}=0$$

$$\left[\frac{1}{r}\frac{\partial V}{\partial r}+\frac{1}{r}\left(\frac{h}{k}-\frac{1}{r}\right)V\right]_{r=r_1}=0$$

$$\left[\frac{\partial V}{\partial r}+\left(\frac{h}{k}-\frac{1}{r_1}\right)V\right]_{r=r_1}=0$$

$$\left[\frac{\partial V}{\partial r}+HV\right]_{r=r_1}=0 \qquad H=\frac{h}{k}-\frac{1}{r_1} \qquad \textit{Robin}$$

Initial-boundary value problem:

$$\frac{\partial^2 V}{\partial r^2}=\frac{1}{\alpha}\frac{\partial V}{\partial t}$$

$$V(r,0)=rU(r,0)=r\left[u_0(r)-u_s(r)\right]$$

$$V\big|_{r=0}=rU\big|_{r=0}=0 \qquad D$$

$$\left[\frac{\partial V}{\partial r}+HV\right]_{r=r_1}=0 \qquad H=\frac{h}{k}-\frac{1}{r_1} \qquad R$$

4) Sturm-Liouville Problem corresponding to the case of Dirichlet-Robin boundary conditions (table SLP):

Eigenvalues λ_n are the positive roots of the equation:

$$\lambda\cos\lambda r_1+H\sin\lambda r_1=0$$

Eigenfunctions $\quad X_n=\sin\lambda_n r$

$$\|X_n\|^2=\frac{r_1}{2}-\frac{\sin(2\lambda_n r_1)}{4\lambda_n}$$

Solution (see Example 2, p.?):

$$V(r,t)=\sum_{n=1}^{\infty}c_n\sin(\lambda_n r)e^{-\alpha\lambda_n^2 t}$$

$$c_n=\frac{\int_0^{r_1} r\left[u_0(r)-u_s(r)\right]\sin(\lambda_n r)\,dr}{\|X_n\|^2}$$

5) Solution:

$$u(r,t)=u_s(r)+\frac{1}{r}V(r,t)$$

6) Example (*turkey-3.mws*) **Roasting of a turkey**

The turkey ($W = 15\ lb$) is assumed to be a sphere with the uniform initial temperature $u_0 = 10\,^oC$. It is exposed to the convective environment at $u_\infty = 150\,^oC$ with the convective coefficient $h = 10\ \frac{W}{m^2 K}$. The turkey is considered to be done when its minimum temperature reaches $u_{done} = 75\,^oC$ (Standard for California). Thermophysical properties of turkey meat used for calculation are from the Table (Section VIII.1.15, p.580).

```
> restart;with(plots):
```

```
> alpha:=0.13e-6;rho:=1050;cp:=3540;k:=0.5;
```

$$\alpha := 0.13\ 10^{-6}$$

$$\rho := 1050$$

$$cp := 3540$$

$$k := 0.5$$

```
> h:=10;
```

$$h := 10$$

```
> qdot:=0.0;
```

$$qdot := 0.$$

```
> W:=15.0;VOL:=W/rho;r1:=fsolve(4/3*Pi*r^3=VOL,r=0..1);
```

$$W := 15.0$$

$$VOL := 0.01428571429$$

$$r1 := 0.1505235493$$

```
> H[2]:=h/k-1/r1;
```

$$H_2 := 13.35652126$$

Specified Temperatures:

```
> uinf:=150;ud:=75;
```

$$uinf := 150$$

$$ud := 75$$

```
> u0:=10;
```

$$u0 := 10$$

Steady State Solution:

```
> us:=qdot*(r1^2-r)/6/k+qdot*r1/3/h+uinf;
```

$$us := 150.$$

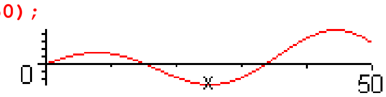
Transient Solution:

characteristic equation:

```
> w(x):=x*cos(x*r1)+H[2]*sin(x*r1);
```

$$w(x) := x\cos(0.1505235493\ x) + 13.35652126\ \sin(0.1505235493\ x)$$

```
> plot(w(x),x=0..50);
```



Eigenvalues:

```
> n:=1: for m from 1 to 20 do y:=fsolve(w(x)=0,x=10*m..10*(m+1)): if type(y,float)
then lambda[n]:=y: n:=n+1 fi od:
> for i to 4 do lambda[i] od;
```

$$15.22059059$$

$$33.80636804$$

$$53.79455908$$

$$74.23157321$$

```
> N:=n-1;
```

$$N := 10$$

Eigenfunctions:

```
> X[n]:=sin(lambda[n]*r);
```

$$X_n := \sin(\lambda_n\, r)$$

```
> NX2[n]:=r1/2-sin(2*lambda[n]*r1)/4/lambda[n];
```

$$NX2_n := 0.07526177465 - \frac{1}{4}\,\frac{\sin(0.3010470986\,\lambda_n)}{\lambda_n}$$

```
> c[n]:=int(r*(u0-us)*X[n],r=0..r1)/NX2[n];
```

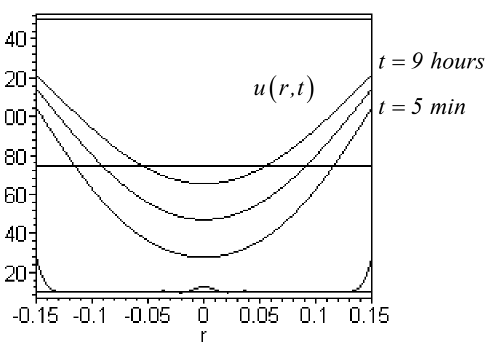
```
> u(r,t):=us+(1/r)*sum(c[n]*X[n]*exp(-alpha*lambda[n]^2*t),n=1..N);
```

$$\begin{aligned} \mathrm{u}(r,t) := 150. + (&-14.93682091\,\sin(15.22059059\,r)\,\mathbf{e}^{(-0.00003011662913\,t)} \\ &+ 4.270369272\,\sin(33.80636804\,r)\,\mathbf{e}^{(-0.0001485731676\,t)} \\ &- 1.825389731\,\sin(53.79455908\,r)\,\mathbf{e}^{(-0.0003762010963\,t)} \\ &+ 0.9848499073\,\sin(74.23157321\,r)\,\mathbf{e}^{(-0.0007163424399\,t)} \\ &- 0.6104851654\,\sin(94.84937974\,r)\,\mathbf{e}^{(-0.001169532629\,t)} \\ &+ 0.4138908063\,\sin(115.5555678\,r)\,\mathbf{e}^{(-0.001735901602\,t)} \\ &- 0.2985392385\,\sin(136.3110699\,r)\,\mathbf{e}^{(-0.002415492011\,t)} \\ &+ 0.2252900588\,\sin(157.0967601\,r)\,\mathbf{e}^{(-0.003208320964\,t)} \\ &- 0.1759525722\,\sin(177.9022284\,r)\,\mathbf{e}^{(-0.004114396373\,t)} \\ &+ 0.1411724164\,\sin(198.7213411\,r)\,\mathbf{e}^{(-0.005133722283\,t)})/r \end{aligned}$$

Solution:

Symmetric Extension:

```
> u2(r,t):=subs(r=-r,u(r,t)):
> t1:=0.5*60*10:t2:=3*60*60:t2:=5*60*60:t3:=7*60*60:t4:=9*60*60:
> z1:=subs(t=t1,u2(r,t)):z2:=subs(t=t2,u2(r,t)):z3:=subs(t=t3,u2(r,t)):z4:=subs(t=t4,u2(r,t)):
> plot({u0,us,ud,z1,z2,z3,z4},r=-r1..r1,color=black,axes=boxed);
```

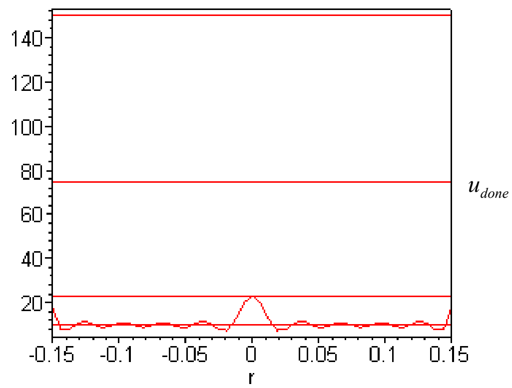


Temperature at the center:

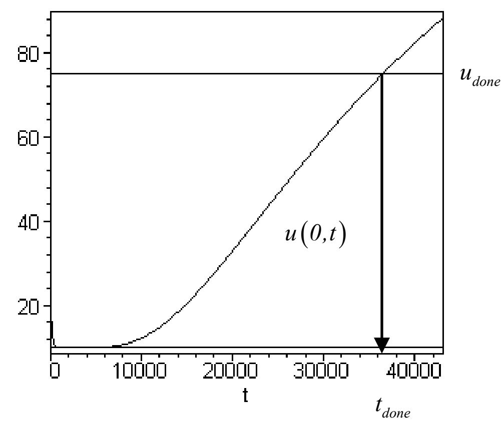
```
> uc:=limit(u2(r,t),r=0);
```

$$\begin{aligned} uc := 150. &- 227.3472358\,\mathbf{e}^{(-0.00003011662913\,t)} + 35.39233832\,\mathbf{e}^{(-0.003208320964\,t)} \\ &+ 144.3656753\,\mathbf{e}^{(-0.0001485731676\,t)} - 31.30235469\,\mathbf{e}^{(-0.004114396373\,t)} \\ &- 98.19603573\,\mathbf{e}^{(-0.0003762010963\,t)} + 28.05397191\,\mathbf{e}^{(-0.005133722283\,t)} \\ &+ 73.10695799\,\mathbf{e}^{(-0.0007163424399\,t)} - 57.90413928\,\mathbf{e}^{(-0.001169532629\,t)} \\ &+ 47.82738713\,\mathbf{e}^{(-0.001735901602\,t)} - 40.69420301\,\mathbf{e}^{(-0.002415492011\,t)} \end{aligned}$$

```
> animate({u2(r,t),uc,ud,u0,us},r=-r1..r1,t=0..11*3600,frames=200,axes=boxed);
```




```
> plot({uc,u0,ud},t=0..12*3600,axes=boxed,color=black);
```

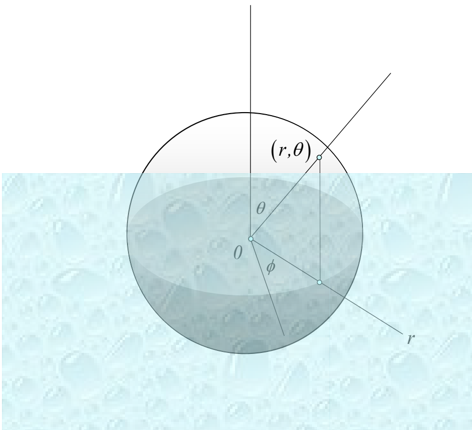


VIII.3.4.2 Heat Equation in Spherical Coordinates: $u(r,\phi,\theta,t)$

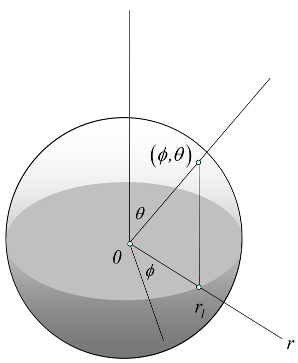
$$\frac{\partial^2 u}{\partial r^2}+2r\frac{\partial u}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\theta^2}+\frac{g}{k}=\frac{1}{\alpha}\frac{\partial u}{\partial t}$$

1) Sphere with angular symmetry $\left(\frac{\partial u}{\partial\phi}=0\right)$: $$\frac{\partial^2 u}{\partial r^2}+2r\frac{\partial u}{\partial r}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\theta^2}+\frac{g}{k}=\frac{1}{\alpha}\frac{\partial u}{\partial t}$$

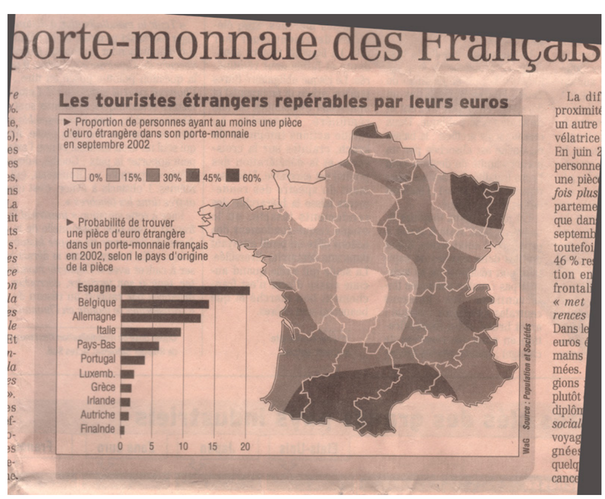
Example: Floating ball



2) Spherical surface of fixed radius r_1 $\left(\frac{\partial u}{\partial r}=0\right)$: $$\frac{1}{r_1^2\sin\theta}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right)+\frac{1}{r_1^2\sin^2\theta}\frac{\partial^2 u}{\partial\theta^2}+\frac{g}{k}=\frac{1}{\alpha}\frac{\partial u}{\partial t}$$



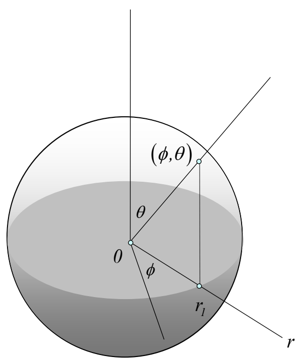
Example: Diffusion of foreign mint coins in France



VIII.3.4.3 Laplace's equation in spherical coordinates

Consider a BVP generated by separation of variables in a PDE in spherical coordinates. We will only see what the Sturm-Liouville problems are in this case.

1. Laplace's Equation



Recall the general form of Laplace's Equation in spherical coordinates for the function $u(r,\phi,\theta),\ \ r\in D$:

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}+\frac{1}{r^2}\frac{\cos\theta}{\sin\theta}\frac{\partial u}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0 \qquad (1)$$

or with differential operators written in self-adjoint form:

$$\frac{\partial}{\partial r}\left[r^2\frac{\partial u}{\partial r}\right]+\frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}+\frac{\cos\theta}{\sin\theta}\frac{\partial u}{\partial\theta}+\frac{\partial^2 u}{\partial\theta^2}=0 \qquad (2)$$

separation of variables

Assume

$$u(r,\varphi,\theta)=R(r)\,\Phi(\varphi)\,\Theta(\theta) \qquad (3)$$

Substitute into equation (1)

$$R''\Phi\Theta+\frac{2}{r}R'\Phi\Theta+\frac{1}{r^2\sin^2\theta}R\Phi''\Theta+\frac{1}{r^2}\frac{\cos\theta}{\sin\theta}R\Phi\Theta'+\frac{1}{r^2}R\Phi\Theta''=0$$

Multiply the equation by $\dfrac{r^2}{R\Phi\Theta}$

$$r^2\frac{R''}{R}+2r\frac{R'}{R}+\frac{1}{\sin^2\theta}\frac{\Phi''}{\Phi}+\frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta}+\frac{\Theta''}{\Theta}=0$$

Consider the axisymmetric case ($\dfrac{\partial}{\partial\phi}=0$):

$$r^2\frac{R''}{R}+2r\frac{R'}{R}+\frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta}+\frac{\Theta''}{\Theta}=0$$

Separate variables and set both sides of the equation equal to the same constant

$$r^2\frac{R''}{R}+2r\frac{R'}{R}=-\frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta}-\frac{\Theta''}{\Theta}=\mu$$

It yields two equations:

1) $\quad r^2\dfrac{R''}{R}+2r\dfrac{R'}{R}=\mu$

which can be rewritten in the form

$r^2R''+2rR'-\mu R=0$ (Euler-Cauchy equation)

or in the self-adjoint Sturm-Liouville form

$$-\frac{1}{1}\left(r^2R'\right)'=\left(-\mu\right)R \qquad (4)$$

Solutions of this equation are sought in the form $R=r^m$

2) $\Theta'' + \frac{\cos\theta}{\sin\theta}\Theta' + \mu\Theta = 0$

Use change of independent variable $x = \cos\theta$, then

$$\Theta' = \frac{d\Theta}{d\theta} = \frac{d\Theta}{dx}\frac{dx}{d\theta} = -\sin\theta\frac{d\Theta}{dx}$$

$$\Theta'' = \frac{d}{d\theta}\left(\frac{d\Theta}{d\theta}\right) = \frac{d}{d\theta}\left(-\sin\theta\frac{d\Theta}{dx}\right) = -\cos\theta\frac{d\Theta}{dx} - \sin\theta\frac{d}{d\theta}\left(\frac{d\Theta}{dx}\right)$$

$$= -\cos\theta\frac{d\Theta}{dx} - \sin\theta\frac{d}{dx}\left(\frac{d\Theta}{dx}\right)\frac{dx}{d\theta} = -\cos\theta\frac{d\Theta}{dx} + \sin^2\theta\frac{d^2\Theta}{dx^2}$$

Substitute into equation

$$-\cos\theta\frac{d\Theta}{dx} + \sin^2\theta\frac{d^2\Theta}{dx^2} - \frac{\cos\theta}{\sin\theta}\sin\theta\frac{d\Theta}{dx} + \mu\Theta = 0$$

$$\sin^2\theta\frac{d^2\Theta}{dx^2} - 2\cos\theta\frac{d\Theta}{dx} + \mu\Theta = 0$$

$$\left(1-x^2\right)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + \mu\Theta = 0$$

or in self-adjoint Sturm-Liouville form:

$$-\frac{d}{dx}\left[\left(1-x^2\right)\frac{d\Theta}{dx}\right] = \mu\Theta \qquad (5)$$

This equation is called Legendre's differential equation. It happens that its solution is bounded only if the separation constant is a non-negative integer of the form

$$\mu = n\left(n+1\right) \qquad n = 0,1,2,...$$

Its solution consists of Legendre polynomials $P_n\left(x\right)$ (see Sec. 5.7).

2. Heat Equation

Consider the axisymmetric heat equation for $u\left(r,t\right)$, $r \in D$, $t > 0$ in spherical coordinates:

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} = a^2\frac{\partial u}{\partial t} \qquad (6)$$

Separation of variables

$$u\left(r,t\right) = R\left(r\right)T\left(t\right)$$

Substitute into equation (6)

$$R''T + \frac{2}{r}R'T = a^2RT'$$

divide by RT and separate variables

$$\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} = a^2\frac{T'}{T} = \mu$$

It yields two ordinary differential equations. Equation for R is

$$r^2R'' + 2rR' - \mu r^2R = 0 \qquad (7)$$

which is a spherical Bessel equation of zero order (see equation (25) in Sec. 5.6 with $n = 0$, AAEM-II).

Eigenvalue problem:

$$LR \equiv \frac{1}{r^2}\left(r^2R'\right)' = \mu R$$

Its solutions are given by spherical Bessel functions

$$j_0\left(r\right) = \sqrt{\frac{\pi}{2}}\frac{J_{1/2}\left(r\right)}{\sqrt{r}}$$

$$y_0\left(r\right) = \sqrt{\frac{\pi}{2}}\frac{Y_{1/2}\left(r\right)}{\sqrt{r}}$$

Why we cannot be completely satisfied with the method of separation of variables?

How about the time dependent boundary conditions, for example?